PROJECTIVE FORMULAS AND UNIFICATION IN LINEAR DISCRETE TEMPORAL MULTI-AGENT LOGICS

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Abstract.

This article continues our studying the problem of unification in multi-agent logics. Based on the approach to the unificational problem through the projective formulas proposed by V. Rybakov and S. Ghilardi, in this paper we consider some linear discrete temporal logics with the agent relations. We proved the projectivity of any unifiable formula in these logics and gave an algorithm for construction the most general unifier.

Keywords: unification, modal temporal logic, passive inference rules, multi-agent relations.

1. Introduction

Unification problems is a popular area in mathematical logic and computer science (cf. for an example [1, 2]). Basic unification problem can be viewed as: whether the formula can be converted into a theorem after replacing the variables. Following closely to the technique from [3], in the papers [4,5] we found a criterions of non-unifiability for formulas in the linear temporal logics of knowledge with multi-modal relations: over the natural numbers ($\mathcal{LTK}$) and over the integers with alternative relations ($\mathcal{LFPK}$) and construct a basis for inference rules passive in these logics.

Unification problem in intuitionistic logic and in propositional modal logics over $K4$ investigated by S. Ghilardi (cf. e.g. [6–8]) via ideas borrowed from projective algebra and technology based on projective formulas. He solved the problem of constructing the finite complete sets of unifiers for a number of considered logics and found efficient algorithms constructing complete sets of unifiers. Such approach proved to be useful and effective in dealing with the admissibility and the basis of admissible rules (cf. Jerabek [9, 10], Iemhoff, Metcalfe [11, 12]): the existence of computable finite sets of unifiers directly implies the solution of the admissibility problem.

Linear temporal logic $\mathcal{LFPL}$ over the integer numbers can be viewed as a linear discrete temporal logic ($\mathcal{LDTL}$, Rybakov [13]), as a particular case of modal logic with linear alternative relations (K. Segerberg [14]) or multi-modal logic (Gabbay et al., [15]). Generally, linear temporal logic models the steps of computational processes and can be effectively used in applications to systems specification, verification [16] and model checking [17].

Solution for admissibility problem for rules in the $\mathcal{LTL}$ (logic over the natural numbers) was proposed by Rybakov [18], basis of admissible rules in $\mathcal{LTL}$ was found by Babenyshev and Rybakov in [19] (without the operator Until. cf. [20]). Solution of the unification problem for $\mathcal{LTL}$ has also been found by Rybakov [21,22]. Particularly, in [21] it is proved that not all unified in $\mathcal{LTL}$ formula are projective;

in [22] the projectivity of any unified formula for the fragment $\mathcal{LTL_U}$ was proved. Last idea, for the such good fragment of $\mathcal{LTL}$, getting only with the operator *Until*, was motivated by the similar result from [23], where it was shown that any formula unifiable in the linear modal logic $\mathcal{S}4.3$ is projective.

Based on the approach and technique proposed in [22], in this paper we consider some linear temporal logics over the integer numbers with the agent relations, for the case with future and past. Here we prove that any formula unifiable in these logics is projective, and hence give an algorithm for construction the most general unifier for any unifiable formula.

2. Semantic deffinitions

We start from notation and basic definitions. Technically we will be based at the technique from [22] for the temporal logic $\mathcal{LTL_U}$ (without the agent relations). For the necessary definitions, notations and semantic for the logic $\mathcal{LFPK}$, we refer reader to [5] (cf. also [20] for the case with no agents).

The alphabet of the language $\mathcal{L}^{LFPK}$ includes a countable set of propositional variables $P := \{p_1, \dots, p_n, \dots\}$, brackets (,) default Boolean logical operations and a variety of unary modal operators $\{\Box_F, \Box_P, \Box_1, \dots, \Box_n\}$. As we can see, the language of the $\mathcal{LFPK}$ extends the language of $\mathcal{LDTL}$ [20] only by the modal operators $\Box_1, \dots, \Box_n$.

Logical operations $\Diamond_F$, $\Diamond_P$, $\Diamond_i$ are defined by means of logical operations $\Box_F$, $\Box_P$, $\Box_i$ in the ordinary way: $\Diamond_F = \neg\Box_F\neg$, $\Diamond_P = \neg\Box_P\neg$, $\Diamond_i = \neg\Box_i\neg$.

The meaning of described modal operations are defined as follows. $\Box_P A$: A is true at all previous and at the current time point; $\Box_F A$: A is true at the given time point and will be true at all future ones. $\Box_i A$ means that A is true at all informational states which available to the agent i.

Semantically, our logic is defined on the Kripke frames and models of linear and discrete stream of the computational process, in which each point in time (world) is associated with a integer number $n \in \mathbb{Z}$.

Definition 1. *LFPK-frame is a temporal $(n+2)$-modal Kripke-frame*

$$T = \langle Z_T, R_F, R_P, R_1, \dots, R_n \rangle,$$

where $R_P = R_F^{-1}$ and:

a. Z_T is the disjoint union of clusters of agents C^t, $t \in \mathbb{Z}$, and $C^{t_1} \bigcap C^{t_2} = \emptyset$ if $t_1 \neq t_2$.

b. $\forall t_1, t_2 \in Z$, if $t_1 \leq t_2$ then $\forall a \in C^{t_1}, \forall b \in C^{t_2}$ $(aR_F b)$ and $(bR_P a)$. None other relations via R_P and R_F are allowed.

c. R_1, …, R_n are some equivalence relations in each separate cluster C^t.

We refer to any such frame as to LFPK-frame.

Frames of this class model situations in which each agent has some information in the current temporary state C^t. Any temporary state C^t consists of a set of information points available at t. The relations R_F and R_P are time connections on a linear stream of information points, wherein for two points w and z term $wR_F z$ means that either w and z are available at the time t, or z will be available in future in subsequent time for w. Conversely, term $wR_P z$ means that either w and z are also available at the same time t, or z was available at previous time w.r.t. w. Each relation R_i, $i = 1, \ldots, n$, reflects the information points available to a particular agent i in the current time state.

Definition 2. *Model M_T on a LFPK-frame T is a tuple $M_T = \langle T, V \rangle$, where V is a valuation of a set of propositional letters $p \in P$ on T, i.e $\forall p \in P$ $[V(p) \subseteq Z_T]$. Given a model $M_T = \langle T, V \rangle$, where T is a LFPK-frame Z_T. Then $\forall w \in Z_T$:*

a. $\langle T, w \rangle \Vdash_V p \Leftrightarrow w \in V(p)$;
b. $\langle T, w \rangle \Vdash_V \Box_F \phi \Leftrightarrow \forall z \in Z_T (w R_F z \Rightarrow \langle T, z \rangle \Vdash_V \phi)$;
c. $\langle T, w \rangle \Vdash_V \Box_P \phi \Leftrightarrow \forall z \in Z_T (w R_P z \Rightarrow \langle T, z \rangle \Vdash_V \phi)$;
d. $\forall i \in I, \langle T, w \rangle \Vdash_V \Box_i \phi \Leftrightarrow \forall z \in Z_T (w R_i z \Rightarrow \langle T, z \rangle \Vdash_V \phi)$;
e. $\langle T, w \rangle \Vdash_V \phi \vee \psi \Leftrightarrow [(\langle T, w \rangle \Vdash_V \phi) \vee (\langle T, w \rangle \Vdash_V \psi)]$;
f. $\langle T, w \rangle \Vdash_V \phi \wedge \psi \Leftrightarrow [(\langle T, w \rangle \Vdash_V \phi) \wedge (\langle T, w \rangle \Vdash_V \psi)]$;
g. $\langle T, w \rangle \Vdash_V \phi \to \psi \Leftrightarrow [(\langle T, w \rangle \Vdash_V \psi) \vee \neg (\langle T, w \rangle \Vdash_V \phi)]$;
h. $\langle T, w \rangle \Vdash_V \neg \phi \;\Leftrightarrow\; [\neg (\langle T, w \rangle \Vdash_V \phi)]$.

Definition 3. *Temporal Linear Future/Past logic of agents knowledge $\mathcal{LFPK}$ is the set of all LFPK formulas valid (true) on all LFPK-frames:*

$$\mathcal{LFPK} := \{A \in Fma(\mathcal{L}^{LFPK}) \mid \forall T, \text{ where } T \text{ is an } LFPK\text{-frame}, (T \Vdash A)\}.$$

The language of the logic $\mathcal{LFPK}^{\mathcal{U}_+}_{\mathcal{U}_-}$ extends the language of $\mathcal{LFPK}$ by the binary operations U_+ and U_-, until operations for future and past.

Definition 4. *To recall action of until logical operations: given a model $M_T = \langle T, V \rangle$, where T is a LFPK-frame Z_T. Then $\forall w \in Z_T$:*

(i) $\langle T, w \rangle \Vdash_V \alpha U_+ \beta \Leftrightarrow \exists j (w R_F j) \Big[\langle T, j \rangle \Vdash_V \beta \,\&\, \forall k : \Big(w R_F k \& \neg (j R_F k) \Rightarrow \langle T, k \rangle \Vdash_V \alpha \Big) \Big]$;

(ii) $\langle T, w \rangle \Vdash_V \alpha U_- \beta \Leftrightarrow \exists j (w R_P j) \Big[\langle T, j \rangle \Vdash_V \beta \,\&\, \forall k : \Big(w R_P k \& \neg (j R_P k) \Rightarrow \langle T, k \rangle \Vdash_V \alpha \Big) \Big]$.

Definition 5. *Temporal Linear Future/Past logic of agents knowledge $\mathcal{LFPK}^{\mathcal{U}_+}_{\mathcal{U}_-}$ is the set of all $LFPK^{U_+}_{U_-}$ formulas valid (true) on all LFPK-frames:*

$$\mathcal{LFPK}^{U_+}_{U_-} := \{A \in Fma(\mathcal{L}^{LFPK^{U_+}_{U_-}}) \mid \forall T, \text{ where } T \text{ is an } LFPK\text{-frame}, (T \Vdash A)\}.$$

The logic $\mathcal{LFPK}^{\mathcal{U}_+,\mathcal{N}}_{\mathcal{U}_-,\mathcal{P}}$ is also the kind of $\mathcal{LFPK}$ with additional binary operations U_+ U_-, as earlier, and a pair of unary operators N and P. The formula $N\phi$ has meaning: ϕ holds in the next time point; $P\phi$ means: ϕ holds on the previous time point.

Definition 6. *To recall action of next and previous logical operations: given a model $M_T = \langle T, V \rangle$, where T is a LFPK-frame Z_T. Then $\forall w \in Z_T$:*

(1) $\langle T, w \rangle \Vdash_V N\alpha \Leftrightarrow \forall z \in Z_T (w \in C^i \& z \in C^{i+1}) \Rightarrow \langle T, z \rangle \Vdash_V \alpha)$;
(2) $\langle T, w \rangle \Vdash_V P\alpha \Leftrightarrow \forall z \in Z_T (w \in C^i \& z \in C^{i-1}) \Rightarrow \langle T, z \rangle \Vdash_V \alpha)$.

Definition 7. *Temporal Linear Future/Past logic of agents knowledge $\mathcal{LFPK}^{\mathcal{U}_+,\mathcal{N}}_{\mathcal{U}_-,\mathcal{P}}$ is the set of all $LFPK^{U_+,N}_{U_-,P}$ formulas valid (true) on all LFPK-frames:*

$$\mathcal{LFPK}^{U_+,N}_{U_-,P} := \{A \in Fma(\mathcal{L}^{LFPK^{U_+,N}_{U_-,P}}) \mid \forall T,\ \textit{where } T \textit{ is an } LFPK\textit{-frame}(T \Vdash A)\}.$$

3. Main results

In this part of the article, we use the notation $\mathcal{L}$ for all considered above logics.

Let $For_{\mathcal{L}}$ is a set of all formulas in the language of $\mathcal{L}$, $P \subseteq For_{\mathcal{L}}$ is a set of letters. A substitution for P is a mapping ϵ of P into For_L. For any such substitution ϵ we may extend it for formulas as follows: $\epsilon(\phi(x_1, \ldots, x_n)) := \phi(\epsilon(x_1), \ldots, \epsilon(x_n))$.

Definition 8. *A formula ϕ is unifiable in $\mathcal{L}$ if there is exist a substitution ϵ (named unifier), s.t. $\epsilon(\phi) \in \mathcal{L}$.*

Definition 9. *A unifier ϵ for ϕ in $\mathcal{L}$ is more general than an unifier ϵ_1 iff there exists a substitution ϵ_2, s.t. for any letter x: $[\epsilon_1(x) \equiv \epsilon_2(\epsilon(x))] \in \mathcal{L}$.*

Definition 10. *A set of unifiers CU for a given formula ϕ in $\mathcal{L}$ is a complete set of unifiers, if for any unifier σ for ϕ in $\mathcal{L}$ there is an unifier σ_1 from CU, where σ_1 is more general than σ.*

Definition 11. *A formula ϕ is called projective in a logic $\mathcal{L}$, if the following holds. There is a substitution σ (projective substitution) such that $\Box_F\Box_P\phi \to [x_i \equiv \sigma(x_i)] \in \mathcal{L}$ for any letter x_i from ϕ, and σ is a unifier for ϕ.*

Lemma 1. *If a substitution σ_p is projective for a formula ϕ in a logic $\mathcal{L}$, then $\{\sigma_p\}$ is a complete set of unifiers for ϕ (i.e. σ_p is most general unifier for ϕ).*

Proof. Indeed, let σ be a unifier for ϕ in $\mathcal{L}$. Since we assume that σ_p is projective for ϕ in $\mathcal{L}$, we have $\Box_F\Box_P\phi \to [x_i \equiv \sigma_p(x_i)] \in \mathcal{L}$ for any letter x_i from ϕ. Acting by σ on the formula above we get $\sigma(\Box_F\Box_P\phi) \to [\sigma(x_i) \equiv \sigma(\sigma_p(x_i))] \in \mathcal{L}$, that is $[\sigma(x_i) \equiv \sigma(\sigma_p(x_i))] \in \mathcal{L}$. □

We now prove the main result of this part of the article for the logic $\mathcal{LFPK}$.

Theorem 1. *Any formula unifiable in $\mathcal{LFPK}$ is projective.*

Proof. For verification the unifiability of any given formula it is enough to consider all possible replacements all it's letters with $\top$, $\bot$, and we will get a unifier for this formula if one exists. So we need to check unifiability in $\mathcal{LFPK}$ only by such possible ground unifiers.

Take any unifiable formula $\phi(x_1, \ldots, x_n)$ in $\mathcal{LFPK}$, and let substitution σ_1, where $\sigma_1(x_i) := g_i$, be a ground unifier for $\phi(x_1, \ldots, x_n)$: $(g_i \in \{\top, \bot\})$ and $\phi(\sigma_1(x_1), \ldots, \sigma_1(x_n)) \in \mathcal{LFPK}$.

Due to unifiability of ϕ in $\mathcal{LFPK}$ by the assumption of the proof, ϕ is true on the model $M_0 := \langle F(1), V\rangle$, where $F(1)$ is a single world frame, with a special valuation V (because there is a certain ground unifier with $\{\top, \bot\}$). It is useful to note here that, despite of the single-world structure of this frame, agents relations set here may be set up by the definition. We define formulas $T(x_i)$ as follows: if $V(x_i) = \oslash$, we set $T(x_i) := \bot$, otherwise $T(x_i) := \top$. For any letter x_i from ϕ we define the following substitution:

$$\sigma(x_i) := \Big(\Box_F\Box_P\phi(x_1, \ldots, x_n) \wedge x_i\Big) \vee \Big(\neg\Box_F\Box_P\phi(x_1, \ldots, x_n) \wedge T(x_i)\Big).$$

The substitution σ is projective for ϕ because of the first disjunct from $\sigma(x_i)$. Indeed, it is clear that $\Box_F\Box_P\phi \to [x_i \equiv \sigma(x_i)] \in \mathcal{LFPK}$ for any letter x_i from ϕ. Now lets show that the substitution σ is a unifier for ϕ. To check it, we take an arbitrary model M_1 for $\mathcal{LFPK}$ and to fix an element w_1 with the valuation V_1 for all letters from ϕ.

If $w_1 \Vdash_{V_1} \Box_F\Box_P\phi$, then everywhere on the frame holds ϕ, hence, the second disjunct of $\sigma(x_i)$ is always disproved in this model. Therefore the truth values for x_i w.r.t. V_1 are always the same as for $\sigma(x_i)$ w.t.r. V_1. In this case, we immediately have $w_1 \Vdash_{V_1} \sigma(\phi)$.

If $w_1 \nVdash_{V_1} \Box_F\Box_P\phi$, then somewhere on the frame holds $\neg\phi$. Then for any world w of the model M_1, $(M_1, w) \Vdash_{V_1} \neg\Box_F\Box_P\phi(x_1, \ldots, x_n)$. Therefore by our definition of the formula $\sigma(x_i)$ above, the truth values of all formulas $\sigma(x_i)$ w.r.t. V_1 everywhere in M_1 are the same as for the formula $T(x_i)$ w.r.t. V in the model M_0. Therefore $w_1 \Vdash_{V_1} \sigma(\phi)$.

$\Box$

Theorem 2. *Any formula unifiable in $\mathcal{LFPK}^{\mathcal{U}_+}_{\mathcal{U}_-}$ is projective.*

Proof. The proof scheme of this statement corresponds exactly to the proof of Theorem 1 above. $\Box$

Theorem 3. *Any formula unifiable in $\mathcal{LFPK}^{\mathcal{U}_+,\mathcal{N}}_{\mathcal{U}_-,\mathcal{P}}$ is projective.*

Proof. Reasoning are analogous to the proofs of the Theorem 1 and 2. $\Box$

Based at the proof of every of the theorems 1–3 and the Lemma 1 the substitution σ gives a construction of a most general unifier for any formula ϕ unifiable in corresponding logic: it's enough just write out the formulas $\sigma(x_i)$. This, as we noticed earlier, also gives positive solution to the open problem of recognizing rules admissible for every $\mathcal{L}$.

Of course, not for any modal logic such fact is true. For example, as shown in [24] for $\mathcal{LTL}$ with the relations *Until* and *Next* this statement is disproved:

Example 1. *(see. eg. [21]). The formula $\phi = \Box(\Box x \vee (\neg x \wedge N\Box x))$ is unifiable in $\mathcal{LTL}$, but not projective.*

Proof. Substitution $x \mapsto \top$ is an obvious unifier for ϕ. Suppose now, that ϕ is projective and π is a corresponding projective unifier. Consider the run N_V (starting from 0: $|N_V| := \{0, 1, 2, \ldots\}$).

$$\overset{x}{\bigcirc} \xrightarrow{N} \overset{\neg x}{\bigcirc} \xrightarrow{N} \overset{\Box x}{\bigcirc} \xrightarrow{N} \overset{\Box x}{\bigcirc} \cdots$$

Since $(N_v, 1) \Vdash_V \Box\phi$, then $(N_v, 1) \Vdash_V x \leftrightarrow \pi(x)$. Therefore, notwithstanding either $(N_v, 0) \Vdash_V \pi(x)$ or $(N_v, 0) \Vdash_V \neg\pi(x)$, we have $(N_v, 0) \Vdash_V \neg\Box\pi(x)$ and, at the same time, $(N_v, 0) \Vdash_V \neg N\Box\pi(x)$. Thus, $(N_v, 0) \Vdash_V \neg\pi(\phi)$, hence π cannot be an ϕ-unifier, a contradiction. $\Box$

At the same time, for a certain «good» fragment of this logic — $\mathcal{LTL}_{\mathcal{U}}$ — only with the operator *Until* in [22] proved otherwise.

Such kind of problem associated, in particular, with the absence of the inverse temporal operations for $\Box, \Diamond$ in $\mathcal{LTL}$. That's why in our considering cases, such problem doesn't arise.

Shown results give us algorithms for effective construction of a most general unifier for any formula ϕ unifiable in considered logics (it's enough just write out the formulas $\sigma(x_i)$) and complete sets of unifier . This, as we noted above, also gives positive solution to the problem of admissibility of inference rules.

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